

Exponential moments of first passage times and related quantities for Lévy processes

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Abstract

For a Lévy process on the real line, we provide complete criteria for the finiteness of exponential moments of the first passage time into the interval (r, ∞) , the sojourn time in the interval $(-\infty, r]$, and the last exit time from $(-\infty, r]$. Moreover, whenever these quantities are finite, we derive their respective asymptotic behavior as $r \rightarrow \infty$.

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1 Introduction and main results

Let $X = (X_t)_{t \geq 0}$ denote a Lévy process on the real line, *i.e.*, a stochastically continuous process with independent and stationary increments and $X_0 = 0$. Throughout the paper, we assume that X has paths in the Skorokhod space of real-valued right-continuous functions with finite left limits.

For $r \geq 0$, define the *first passage time* into the interval (r, ∞)

$$T_r := \inf\{t \geq 0 : X_t > r\},$$

with the convention that $\inf \emptyset = \infty$, the *sojourn time* in the interval $(-\infty, r]$

$$N_r := \int_0^\infty \mathbb{1}_{\{X_t \leq r\}} dt,$$

and the *last exit time* from $(-\infty, r]$

$$\varrho_r := \sup\{t \geq 0 : X_t \leq r\}.$$

It can be checked that

$$T_r \leq N_r \leq \varrho_r. \quad (1.1)$$

In the paper at hand, we derive necessary and sufficient conditions for the finiteness of exponential moments of these three quantities and, thus, obtain the analogues of the results obtained by two of the three authors for random

walks [11, 12]. Similar results for power moments have been obtained in [9, 15] by different methods.

Observe that, by the Blumenthal zero-one law, $\mathbb{P}\{T_0 = 0\} \in \{0, 1\}$. In many relevant cases, $T_0 = 0$ a.s., yet $\mathbb{E}[e^{aT_r}] = \infty$ for any $r > 0$. In fact, whether or not $\mathbb{P}\{T_0 = 0\} = 1$ is a small-time property of X (that has been investigated in detail in [14, Theorem 47.5]), whereas we are interested in the long-time behavior of X . Therefore, we focus on exponential moments of T_r for positive r .

Our main results can be summarized as follows: Proposition 1.1 deals with the case when X is a subordinator and gives criteria for the finiteness of exponential moments of T_r , N_r , ϱ_r . The corresponding results in the case when X is not a subordinator are given in Theorems 1.2 and 1.3. Finally, Theorems 1.5 and 1.6 give the asymptotics of the respective exponential moment when $r \rightarrow \infty$. All theorems exclude the case of compound Poisson processes, where – contrary to general Lévy processes – the problem can be completely reduced to the random walk setup [11, 12] (as outlined in Remark 1.4). After stating the main results, their proofs are given in Section 2. We comment on a number of special cases and examples in Section 3.

We further mention that the finiteness of exponential moments of T_r is naturally connected to the asymptotic behavior of persistence probabilities of X , we refer to the recent survey [4] for details.

We first consider the (simple) case when X is a subordinator. The first result is a direct consequence of the corresponding result for renewal sequences.

Proposition 1.1. *Let X be a subordinator with $\mathbb{P}\{X_1 = 0\} < 1$.*

(a) *If X is not a compound Poisson process. Then, for every $a > 0$,*

$$\mathbb{E}[e^{aT_r}] < \infty \quad \text{for all } r \geq 0.$$

(b) *Let X be a compound Poisson process (with positive jumps) of rate $\lambda > 0$. Then, for $a > 0$, the following conditions are equivalent:*

$$\mathbb{E}[e^{aT_r}] < \infty \quad \text{for some (hence every) } r \geq 0; \tag{1.2}$$

$$a < \lambda. \tag{1.3}$$

In both cases the same statements also hold for N_r and ϱ_r .

For $r \geq 0$, let $T_r^1 = \inf\{k \in \mathbb{N}_0 : X_k > r\}$, $N^1(x) := \#\{k \in \mathbb{N}_0 : X_k \leq x\}$ and $\varrho_r^1 = \sup\{k \in \mathbb{N}_0 : X_k \leq r\}$ be the first passage time of the level r , the number of visits to the interval $(-\infty, r]$ and the last exit time from the interval $(-\infty, r]$ by the embedded skeleton-1 random walk $(X_k)_{k \in \mathbb{N}_0}$. Clearly,

$$T_r \leq T_r^1. \tag{1.4}$$

Further, denote by $(L_t^{-1})_{t \geq 0}$ the ascending ladder time process of $(X_t)_{t \geq 0}$, see [5, p. 157] for the precise definition of this process.

Theorem 1.2. *Let $\mathbb{P}\{X_1 < 0\} > 0$ and $a > 0$. Then the following assertions are equivalent:*

$$\mathbb{E}[e^{aT_r}] < \infty \quad \text{for some/every } r > 0; \quad (1.5)$$

$$\mathbb{E}[e^{aN_r}] < \infty \quad \text{for some/every } r \geq 0; \quad (1.6)$$

$$\mathbb{E}[e^{aL_1^{-1}}] < \infty; \quad (1.7)$$

$$V_a(r) := \int_1^\infty e^{at} t^{-1} \mathbb{P}\{X_t \leq r\} dt < \infty \quad \text{for some/every } r \in \mathbb{R}; \quad (1.8)$$

$$\mathbb{E}[e^{aT_r^1}] < \infty \quad \text{for some/every } r \geq 0; \quad (1.9)$$

$$\mathbb{E}[e^{aN_r^1}] < \infty \quad \text{for some/every } r \geq 0; \quad (1.10)$$

$$V_a^1(r) := \sum_{n \geq 1} e^{an} n^{-1} \mathbb{P}\{X_n \leq r\} < \infty \quad \text{for some/every } r \in \mathbb{R}; \quad (1.11)$$

$$a \leq R := -\log \inf_{\theta \geq 0} \varphi(\theta) \quad (1.12)$$

where φ denotes the Laplace transform of X_1 , i.e., $\varphi(\theta) = \mathbb{E}[e^{-\theta X_1}]$, $\theta \geq 0$.

Theorem 1.3. *Let $\mathbb{P}\{X_1 < 0\} > 0$ and $a > 0$. Then the following assertions are equivalent:*

$$\mathbb{E}[e^{ae_r}] < \infty \quad \text{for some/every } r \geq 0; \quad (1.13)$$

$$U_a(r) := \int_0^\infty e^{at} \mathbb{P}\{X_t \leq r\} dt < \infty \quad \text{for some/every } r \in \mathbb{R}; \quad (1.14)$$

$$\mathbb{E}[e^{ae_r^1}] < \infty \quad \text{for some/every } r \geq 0; \quad (1.15)$$

$$U_a^1(r) := \sum_{n \geq 0} e^{an} \mathbb{P}\{X_n \leq r\} < \infty \quad \text{for some/every } r \in \mathbb{R}; \quad (1.16)$$

$$a < R := -\log \inf_{t \geq 0} \varphi(t) \quad \text{or} \quad a = R \text{ and } \mathbb{E}[X_1 e^{-\gamma X_1}] > 0 \quad (1.17)$$

where γ is the unique positive number with $\mathbb{E}[e^{-\gamma X_1}] = e^{-R}$.

Conditions (1.12) and (1.17) can be reformulated in terms of the characteristic exponent of X_1 . For $t \geq 0$, let $\phi_t(\theta) = \mathbb{E}[e^{i\theta X_t}] = \exp(t\Psi(i\theta))$, $\theta \geq 0$ be the characteristic function of X_t where the Lévy exponent Ψ is given by the Lévy-Khintchine formula (see [5, p. 13] or [14, p. 37])

$$\Psi(i\theta) = i\theta\mu + \frac{1}{2}\sigma^2(i\theta)^2 + \int_{\mathbb{R}} (e^{i\theta x} - 1 - i\theta x \mathbb{1}_{[-1,1]}(x)) \Pi(dx) \quad (1.18)$$

where $\mu \in \mathbb{R}$, $\sigma^2 \geq 0$ and Π is a Lévy measure on \mathbb{R} . Henceforth, we denote by φ the Laplace transform of X_1 . Then [14, Theorem 25.17] implies that $\varphi(\theta) = \exp(\Psi(-\theta))$ for every $\theta \geq 0$ where

$$\Psi(-\theta) = -\theta\mu + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (e^{-\theta x} - 1 + \theta x \mathbb{1}_{[-1,1]}(x)) \Pi(dx). \quad (1.19)$$

It is worth stressing that $\varphi(\theta) = \infty$ iff the integral on the right-hand side of (1.19) equals $+\infty$. This is why the identity holds for *every* $\theta \geq 0$. Therefore,

$$\begin{aligned} -\log \inf_{\theta \geq 0} \varphi(\theta) &= \sup_{\theta \geq 0} (-\Psi(-\theta)) \\ &= \sup_{\theta \geq 0} \left(\theta\mu - \frac{1}{2}\sigma^2\theta^2 - \int_{\mathbb{R}} (e^{-\theta x} - 1 + \theta x \mathbb{1}_{[-1,1]}(x)) \Pi(dx) \right). \end{aligned} \quad (1.20)$$

We continue with the asymptotic behavior of $\mathbb{E}[e^{aT_r}]$, $\mathbb{E}[e^{aN_r}]$ and $\mathbb{E}[e^{a\varrho_r}]$ as $r \rightarrow \infty$ in the situations where these quantities are finite. In order to avoid distinguishing between the non-lattice and the lattice case¹ we exclude the latter case from the discussion. What is more, we shall exclude the more general case that X is a compound Poisson process. As Remark 1.4 below shows, this case can be reduced to the random walk setup [11, 12]. Contrary to this, for processes which are not compound Poisson the reduction to random walks does not seem possible and different techniques have to be used.

Remark 1.4. Assume that X is a compound Poisson process. Then there is a Poisson process $(N(t))_{t \geq 0}$ with rate $\lambda > 0$ and a sequence $(Y_k)_{k \in \mathbb{N}}$ of i.i.d. random values independent of $(N(t))_{t \geq 0}$ such that $X_t = S_{N(t)}$, $t \geq 0$ where $S_0 = 0$ and $S_n = \sum_{k=1}^n Y_k$, $n \in \mathbb{N}$. Let $\tau(r)$, $n(r)$, and $\rho(r)$ be the first passage time, number of visits, and last exit time for the random walk $(S_n)_{n \in \mathbb{N}_0}$. Then the moments of T_r , N_r and ϱ_r for the compound Poisson process can be expressed in terms of the respective quantities for the random walk, $\tau(r)$, $n(r)$ and $\rho(r)$, as will be outlined below.

First notice that $a < \lambda$ is necessary for any of the three exponential moments to be finite, which follows from $\mathbb{P}\{T_r > t\} \geq \mathbb{P}\{N(t) = 0\} = e^{-\lambda t}$ and (1.1). We can thus define $e^b := \lambda/(\lambda - a)$. Then the crucial equations read

$$\mathbb{E}[e^{aT_r}] = \mathbb{E}[e^{b\tau(r)}], \quad \mathbb{E}[e^{aN_r}] = \mathbb{E}[e^{bn(r)}] \quad \text{and} \quad \mathbb{E}[e^{a\varrho_r}] = \mathbb{E}[e^{b\rho(r)}]$$

meaning that, for each of these equations, when one side of the equation is finite, then so is the other and the two sides coincide.

Before we state the results describing the asymptotic behavior of $\mathbb{E}[e^{aT_r}]$, $\mathbb{E}[e^{aN_r}]$ and $\mathbb{E}[e^{a\varrho_r}]$ as $r \rightarrow \infty$, we remind the reader of the exponential change of measure known as the Esscher transform. Here and throughout the paper, whenever $0 < a \leq R = -\log \inf_{\theta \geq 0} \varphi(\theta)$ and $\mathbb{P}\{X_1 < 0\} > 0$, we write γ for the minimal $\gamma > 0$ satisfying

$$\varphi(\gamma) = \mathbb{E}[e^{-\gamma X_1}] = e^{-a}. \quad (1.21)$$

It can be checked that $(e^{-\gamma X_t + at})_{t \geq 0}$ is a unit-mean martingale with respect to $\mathcal{F} := (\mathcal{F}_t)_{t \geq 0}$ where, for each $t \geq 0$, \mathcal{F}_t is the completion of $\mathcal{F}_t^0 := \sigma(X_s : 0 \leq s \leq t)$. This allows to define a new probability measure \mathbb{P}^γ by

$$\left. \frac{d\mathbb{P}^\gamma}{d\mathbb{P}} \right|_{\mathcal{F}_t} = e^{-\gamma X_t + at}, \quad t \geq 0. \quad (1.22)$$

¹ The Lévy process X is called lattice if, for some $d > 0$, $\mathbb{P}\{X_t \in d\mathbb{Z}\} = 1$ for all $t \geq 0$.

From [13, Theorem 3.9] we conclude that under \mathbb{P}^γ , X still is a Lévy process with Laplace transform

$$\mathbb{E}^\gamma[e^{-\theta X_1}] = e^a \mathbb{E}[e^{-(\gamma+\theta)X_1}] = e^a \varphi(\gamma + \theta), \quad \theta \geq 0.$$

Since $\mathbb{E}^\gamma[X_1] = -e^a \varphi'(\gamma)$ (where φ' denotes the left derivative of φ) and since φ is decreasing and convex on $[0, \gamma]$, there are only two possibilities:

$$\text{Either } \mathbb{E}^\gamma[X_1] \in (0, \infty) \quad \text{or} \quad \mathbb{E}^\gamma[X_1] = 0. \quad (1.23)$$

When $a < R$, then the first alternative in (1.23) prevails. When $a = R$, then typically $\varphi'(\gamma) = 0$ since then γ is the unique minimizer of φ on $[0, \infty)$. But even if $a = R$ it can occur that $\mathbb{E}^\gamma[X_1] > 0$ or, equivalently, $\varphi'(\gamma) < 0$.

Theorem 1.5. *Assume that $\mathbb{P}\{X_1 < 0\} > 0$ and that X is not a compound Poisson process. Further, let $a > 0$ and suppose that the equivalent conditions of Theorem 1.2 hold. Then there is a minimal $\gamma > 0$ such that $\mathbb{E}[e^{-\gamma X_1}] = e^{-a}$.*

(a) *We have $\mathbb{E}^\gamma[X_{L_1^{-1}}] \in (0, \infty)$, and*

$$\lim_{r \rightarrow \infty} e^{-\gamma r} \mathbb{E}[e^{aT_r}] = \frac{\log \mathbb{E}[e^{aL_1^{-1}}]}{\gamma \mathbb{E}^\gamma[X_{L_1^{-1}}]}. \quad (1.24)$$

(b) *With $g(x) := e^{\gamma x} \mathbb{E}[e^{aN(-x)}]$, $x \geq 0$ and H denoting a random variable being the distributional limit of the overshoot $X_{T_r} - r$ as $r \rightarrow \infty$ under \mathbb{P}^γ , it holds that*

$$\lim_{r \rightarrow \infty} e^{-\gamma r} \mathbb{E}[e^{aN_r}] = \mathbb{E}^\gamma[g(H)] \in (0, \infty). \quad (1.25)$$

Theorem 1.6. *Assume that $\mathbb{P}\{X_1 < 0\} > 0$ and that X is not a compound Poisson process. Further, let $a > 0$ and suppose that the equivalent conditions of Theorem 1.3 hold. Then there exists a minimal $\gamma > 0$ such that $\mathbb{E}[e^{-\gamma X_1}] = e^{-a}$ which additionally satisfies $\mathbb{E}[X_1 e^{-\gamma X_1}] \in (0, \infty)$. Moreover,*

$$\lim_{r \rightarrow \infty} e^{-\gamma r} U_a(r) = \lim_{r \rightarrow \infty} e^{-\gamma r} \int_0^\infty e^{at} \mathbb{P}\{X_t \leq r\} dt = \frac{e^{-a}}{\gamma \mathbb{E}[X_1 e^{-\gamma X_1}]} \quad (1.26)$$

and

$$\lim_{r \rightarrow \infty} e^{-\gamma r} \mathbb{E}[e^{a\varrho_r}] = e^{\gamma r} \cdot \frac{ae^{-a} \mathbb{E}[e^{-\gamma \inf_{t \geq 0} X_t}]}{\gamma \mathbb{E}[X_1 e^{-\gamma X_1}]}. \quad (1.27)$$

Here, $\mathbb{E}[e^{-\gamma \inf_{t \geq 0} X_t}] < \infty$.

2 Proofs of the main results

Proof of Proposition 1.1. The proposition follows from the corresponding result for random walks [12, Theorem 2.1] and the following three observations:

- (i) $T_r \leq T_r^1 \leq T_r + 1$ for all $r \geq 0$ since X has nondecreasing paths a.s.;
- (ii) $\mathbb{P}\{X_1 = 0\} = e^{-\lambda}$ when X is a compound Poisson process with rate λ , and $\mathbb{P}\{X_1 = 0\} = 0$, otherwise.
- (iii) $T_r = N_r = \varrho_r$ when X is a subordinator. □

Before we give the proofs of Theorems 1.2 and 1.3, we provide a short technical interlude. For all $r, t > 0$, by definition, we have

$$\{T_r > t\} \subseteq \{\sup_{0 \leq s \leq t} X_s \leq r\} \subseteq \{T_r \geq t\}. \quad (2.1)$$

Since $\mathbb{P}\{T_r > t\} \neq \mathbb{P}\{T_r \geq t\}$ for at most countably many $t > 0$, we conclude:

$$\frac{1}{a} \mathbb{E}[e^{aT_r} - 1] = \int_0^\infty e^{at} \mathbb{P}\{T_r > t\} dt = \int_0^\infty e^{at} \mathbb{P}\{\sup_{0 \leq s \leq t} X_s \leq r\} dt. \quad (2.2)$$

Turning to ϱ_r , notice that, since X has càdlàg paths, for all $r, t \geq 0$,

$$\{\inf_{s \geq t} X_s < r\} \subseteq \{\varrho_r > t\} \subseteq \{\inf_{s \geq t} X_s \leq r\}. \quad (2.3)$$

Proof of Theorem 1.2. Let $a > 0$. From the corresponding results for the embedded zero-delayed random walk $(X_n)_{n \in \mathbb{N}_0}$, see [11, Theorem 1.2], we infer the equivalence of (1.9), (1.10), (1.11) and (1.12). The only detail that needs clarification is that in [11, Theorem 1.2], convergence of the series $V_a^1(r)$ is considered only for $r \geq 0$. However, the fact that $V_a^1(r)$ (resp. $V_a(r)$) is finite for some $r \in \mathbb{R}$ if and only if it is finite for all $r \in \mathbb{R}$ follows from an application of the Esscher transform and standard arguments.

Further, (1.9) implies (1.5) by (1.4), and (1.6) implies (1.5) by (1.1).

Now we show that (1.5) implies (1.9). To this end, assume that, for some $r > 0$, $\mathbb{E}[e^{aT_r}] < \infty$. By (2.2), there is a $t > 0$ with $\mathbb{P}\{\sup_{0 \leq s \leq t} X_s \leq r\} > 0$. Since X is not a subordinator, there is an $\epsilon > 0$ with $\mathbb{P}\{X_t \leq -\epsilon\} > 0$. The random variables X_t and $\sup_{0 \leq s \leq t} X_s$ are associated (see [10] for the definition and fundamental properties of association), thus

$$\mathbb{P}\{\sup_{0 \leq s \leq t} X_s \leq r, X_t \leq -\epsilon\} \geq \mathbb{P}\{\sup_{0 \leq s \leq t} X_s \leq r\} \mathbb{P}\{X_t \leq -\epsilon\} > 0.$$

The Markov property at time t thus yields

$$\begin{aligned} \mathbb{E}[e^{aT_r}] &\geq \mathbb{E}[e^{aT_r} \mathbb{1}_{\{\sup_{0 \leq s \leq t} X_s \leq r, X_t \leq -\epsilon\}}] \\ &\geq \mathbb{P}\{\sup_{0 \leq s \leq t} X_s \leq r, X_t \leq -\epsilon\} e^{at} \mathbb{E}[e^{aT_{r+\epsilon}}], \end{aligned}$$

in particular, $\mathbb{E}[e^{aT_{r+\epsilon}}] < \infty$. Consequently, $\mathbb{E}[e^{aT_r}] < \infty$ for all $r > 0$. Now, for $r > 0$, define $\tilde{T}_r := \inf\{t \geq 1 : X_t > r\}$. We claim that $\mathbb{E}[e^{a\tilde{T}_r}] < \infty$ for all $r > 0$. Indeed, for any $t > 1$, by [3, Lemma 12], which makes use of the fact that $\sup_{0 \leq s \leq 1} X_s$ and $\sup_{1 \leq s \leq t} X_s$ are associated random variables, we infer

$$\mathbb{P}\{\sup_{0 \leq s \leq t} X_s \leq r\} \geq \mathbb{P}\{\sup_{0 \leq s \leq 1} X_s \leq r\} \mathbb{P}\{\sup_{1 \leq s \leq t} X_s \leq r\}.$$

Since the paths of X are locally bounded, $\mathbb{P}\{\sup_{0 \leq s \leq 1} X_s \leq r\} > 0$ for all sufficiently large $r > 0$. For any such r , using the analogue of (2.2) for \tilde{T}_r instead of T_r , we infer

$$\begin{aligned} \mathbb{E}[e^{a\tilde{T}_r}] &= 1 + \int_0^\infty a e^{at} \mathbb{P}\{\tilde{T}_r > t\} dt = e^a + \int_1^\infty a e^{at} \mathbb{P}\left\{\sup_{1 \leq s \leq t} X_s \leq r\right\} dt \\ &\leq e^a + \mathbb{P}\left\{\sup_{0 \leq s \leq 1} X_s \leq r\right\}^{-1} \int_1^\infty a e^{at} \mathbb{P}\left\{\sup_{0 \leq s \leq t} X_s \leq r\right\} dt \\ &\leq e^a + \mathbb{P}\left\{\sup_{0 \leq s \leq 1} X_s \leq r\right\}^{-1} \mathbb{E}[e^{aT_r}]. \end{aligned}$$

In particular, $\mathbb{E}[e^{a\tilde{T}_r}] < \infty$ for all sufficiently large $r > 0$ and hence for all $r > 0$. Now fix $r > 0$. We show that $\mathbb{E}[e^{aT_r^1}] < \infty$. To this end, for $s \in \mathbb{R}$, define $A_s := \{\inf_{\tilde{T}_{r-1} \leq t \leq \tilde{T}_r} X_t \leq r + s\}$. We can choose s small enough to ensure

$$\gamma_s := \mathbb{E}[e^{a\tilde{T}_r} \mathbb{1}_{A_s}] < 1.$$

We assume without loss of generality that $s \leq 0$. Let

$$T_r^{(n)} = \begin{cases} T_{1-s} & \text{for } n = 0, \\ \inf\{t \geq T_r^{(n-1)} + 1 : X_t - X_{T_r^{(n-1)}} > r\} & \text{for } n = 1, 2, \dots \end{cases}$$

By the strong Markov property, $(T_r^{(n)} - T_r^{(n-1)}, (X_{T_r^{(n-1)}+t} - X_{T_r^{(n-1)}})_{0 \leq t \leq T_r^{(n)} - T_r^{(n-1)}})$, $n \in \mathbb{N}$ are independent copies of $(\tilde{T}_r, (X_t)_{0 \leq t \leq \tilde{T}_r})$ and independent of T_{1-s} . In particular, the $T_r^{(n)} - T_r^{(n-1)}$, $n \in \mathbb{N}$ have a finite exponential moment of order $a > 0$. Define

$$\sigma := \inf\{n \in \mathbb{N} : \inf_{T_r^{(n-1)} \leq t \leq T_r^{(n)}} (X_t - X_{T_r^{(n-1)}}) > r + s\}.$$

By construction, $X_{\lfloor T_r^{(\sigma)} \rfloor} > r + s + X_{T_r^{(\sigma-1)}} > r$, hence $T_r^1 \leq T_r^{(\sigma)}$. Further, with $A_k := \{\inf_{T_r^{(k-1)} \leq t \leq T_r^{(k)}} (X_t - X_{T_r^{(k-1)}}) \leq r + s\}$, we have

$$\{\sigma = n\} = A_1 \cap \dots \cap A_{n-1} \cap A_n^c.$$

Consequently,

$$\begin{aligned} \mathbb{E}[e^{aT_r^1}] &\leq \mathbb{E}[e^{aT_r^{(\sigma)}}] = \sum_{n \geq 1} \mathbb{E}\left[\mathbb{1}_{\{\sigma=n\}} e^{aT_{1-s}} \prod_{k=1}^n e^{a(T_r^{(k)} - T_r^{(k-1)})}\right] \\ &= \sum_{n \geq 1} \mathbb{E}\left[e^{aT_{1-s}} \mathbb{1}_{A_n^c} e^{a(T_r^{(n)} - T_r^{(n-1)})} \prod_{k=1}^{n-1} \mathbb{1}_{A_k} e^{a(T_r^{(k)} - T_r^{(k-1)})}\right]. \end{aligned}$$

Since A_k and A_k^c are measurable with respect to the σ -field generated by $(T_r^{(k)} - T_r^{(k-1)}, (X_{T_r^{(k-1)}+t} - X_{T_r^{(k-1)}})_{0 \leq t \leq T_r^{(k)} - T_r^{(k-1)}})$, $k \in \mathbb{N}$, the factors $e^{aT_{1-s}}$, $\mathbb{1}_{A_n^c} e^{a(T_r^{(n)} - T_r^{(n-1)})}$ and $\mathbb{1}_{A_k} e^{a(T_r^{(k)} - T_r^{(k-1)})}$, $k = 1, \dots, n-1$, are independent. Thus, we further conclude

$$\begin{aligned} \mathbb{E}[e^{aT_r^1}] &\leq \sum_{n \geq 1} \mathbb{E}[e^{aT_{1-s}}] \mathbb{E}\left[\mathbb{1}_{A_n^c} e^{a(T_r^{(n)} - T_r^{(n-1)})}\right] \prod_{k=1}^{n-1} \mathbb{E}\left[\mathbb{1}_{A_k} e^{a(T_r^{(k)} - T_r^{(k-1)})}\right] \\ &= \mathbb{E}[e^{aT_{1-s}}] \mathbb{E}\left[e^{a\tilde{T}_r} \mathbb{1}_{A_s^c}\right] \sum_{n \geq 1} \gamma_s^{n-1} < \infty. \end{aligned}$$

For the proof of the equivalence of (1.8) and (1.11) set $I_n := \inf_{n-1 \leq t \leq n} X_t - X_{n-1}$ and $S_n := \sup_{n-1 \leq t \leq n} X_t - X_{n-1}$, $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, I_n and S_n are independent of X_{n-1} and have the same distribution as I_1 and S_1 , respectively. Now observe that, for $t > 1$ and $n \in \mathbb{N}$ such that $n \leq t \leq n+1$

$$\frac{1}{2} \frac{e^{an}}{n} \mathbb{P}\{X_n + S_{n+1} \leq r\} \leq \frac{e^{at}}{t} \mathbb{P}\{X_t \leq r\} \leq e^a \frac{e^{an}}{n} \mathbb{P}\{X_n + I_{n+1} \leq r\}.$$

Integrating over $t \in (1, \infty)$ leads to

$$\frac{1}{2}\mathbb{E}[V_a^1(r - S_1)] \leq \int_1^\infty \frac{e^{at}}{t} \mathbb{P}\{X_t \leq r\} dt \leq e^a \mathbb{E}[V_a^1(r - I_1)].$$

Now assume that (1.8) holds for some $r \in \mathbb{R}$. Then $\mathbb{E}[V_a^1(r - S_1)] < \infty$ and, in particular, there is some $x > 0$ with $V_a^1(-x) < \infty$. This implies (1.11) since $V_a^1(y) < \infty$ for some $y \in \mathbb{R}$ if and only if $V_a^1(y) < \infty$ for all $y \in \mathbb{R}$.

Conversely, when (1.11) holds, then $V_a^1(r) < \infty$ for all $r > 0$. (1.8) follows if we can prove that $\mathbb{E}[V_a^1(r - I_1)] < \infty$. By the already established equivalence between (1.11) and (1.12), we know that $a \leq R$. When (1.17) holds, then, by Proposition 5.1 in [2], there exists a constant $C > 0$ such that

$$V_a^1(x) \leq Ce^{\gamma x} \quad \text{for all } x \geq 0 \quad (2.4)$$

where $\gamma > 0$ is the minimal root of the equation $\varphi(\gamma) = e^{-a}$. In particular, $\mathbb{E}[V_a(r - I_1)] \leq Ce^{\gamma r} \mathbb{E}[e^{-\gamma I_1}]$ and the latter expectation is finite due to Lemma A.1 in the appendix. It remains to deal with the case when (1.12) holds but (1.17) fails, that is, the case when $a = R$ and $\mathbb{E}[X_1 e^{-\gamma X_1}] = 0$. We claim that (2.4) holds in this case, too. Once the claim is proved, (1.11) follows as in the previous case. To prove the claim, we use an exponential change of measure to conclude that

$$V_a^1(x) = \sum_{n \geq 1} \frac{e^{an}}{n} \mathbb{P}\{X_n \leq x\} = \sum_{n \geq 1} \frac{1}{n} \mathbb{E}^\gamma[e^{\gamma X_n} \mathbb{1}_{\{X_n \leq x\}}] \quad (2.5)$$

$$= e^{\gamma x} \int e^{-\gamma(x-y)} \mathbb{1}_{[0, \infty)}(x-y) V^{1, \gamma}(dy) \quad (2.6)$$

where $V^{1, \gamma}(dy) = \sum_{n \geq 1} \frac{1}{n} \mathbb{P}^\gamma\{X_n \in dy\}$ denotes the harmonic renewal measure of the random walk $(X_n)_{n \in \mathbb{N}_0}$ under \mathbb{P}^γ , which is centered in the given situation. Hence, we conclude from [1, Theorem 1.3] that $V^{1, \gamma}$ is locally finite. Moreover, $V^{1, \gamma}$ is uniformly locally bounded since, for $a < b$, by the strong Markov property at $\tau_{[a, b]} := \inf\{n \in \mathbb{N} : X_n \in [a, b]\}$,

$$\begin{aligned} V^{1, \gamma}([a, b]) &\leq \mathbb{E}^\gamma \left[\mathbb{1}_{\{\tau_{[a, b]} < \infty\}} \sum_{n \geq \tau_{[a, b]}} \frac{1}{n} \mathbb{1}_{\{|X_n - X_{\tau_{[a, b]}}| \leq b-a\}} \right] \\ &\leq \mathbb{P}^\gamma\{\tau_{[a, b]} < \infty\} V^{1, \gamma}([-(b-a), b-a]). \end{aligned}$$

Consequently, the integral in (2.6) remains bounded as $x \rightarrow \infty$ and (2.4) follows.

Next, we show that (1.8) implies (1.6). According to the already proved equivalence between (1.8) and (1.12) we can assume that (1.8) holds for every $r \geq 0$, particularly for $r = 0$. By Sparre-Anderson's identity [5, Lemma 15 on p. 170], N_0 has the same law as $G := \sup\{t \geq 0 : X_t = \inf_{0 \leq s \leq t} X_s\}$, the last zero of the process reflected at the infimum. Letting $q \downarrow 0$ and using the monotone convergence theorem in [5, Eq. (VI.5)], we infer

$$\mathbb{E}[e^{-\theta G}] = \exp \left(- \int_0^\infty (1 - e^{-\theta t}) t^{-1} \mathbb{P}\{X_t \leq 0\} dt \right), \quad \theta \geq 0.$$

This shows that G has an infinitely divisible law with Lévy measure $\nu(dt) = \mathbb{1}_{(0,\infty)}(t)t^{-1}\mathbb{P}\{X_t \leq 0\}dt$. Condition (1.8) with $r = 0$ implies first that it is indeed a Lévy measure because

$$\int_{(0,\infty)} (t \wedge 1) \nu(dt) = \int_0^1 \mathbb{P}\{X_t \leq 0\} dt + \int_1^\infty t^{-1} \mathbb{P}\{X_t \leq 0\} dt < \infty,$$

and second that $\int_{(1,\infty)} e^{at} \nu(dt) < \infty$. An appeal to Theorem 25.17 in [14] gives

$$\mathbb{E}[e^{aN_0}] = \mathbb{E}[e^{aG}] = \exp\left(\int_0^\infty (e^{at} - 1)t^{-1}\mathbb{P}\{X_t \leq 0\} dt\right) < \infty. \quad (2.7)$$

Further, we already know that (1.8) guarantees $\mathbb{E}[e^{aT_r}] < \infty$ for every $r > 0$. Since

$$N_r = T_r + \int_{T_r}^\infty \mathbb{1}_{\{X_t - X_{T_r} \leq r - X_{T_r}\}} dt \leq T_r + \int_{T_r}^\infty \mathbb{1}_{\{X_t - X_{T_r} \leq 0\}} dt,$$

and the last summand is independent of T_r and has the same law as N_0 we infer $\mathbb{E}[e^{aN_r}] \leq \mathbb{E}[e^{aT_r}]\mathbb{E}[e^{aN_0}] < \infty$ for $r > 0$.

We now show that (1.8) implies (1.7). To this end, assume that (1.8) holds and use the already established equivalence between (1.8) and (1.12) to conclude that $0 < a \leq R$. It is known (see *e.g.* [5, p. 166]) that $(L_t^{-1})_{t \geq 0}$ is a subordinator (without killing) with Laplace exponent

$$\begin{aligned} -\log \mathbb{E}[e^{-\theta L_1^{-1}}] &= c \exp\left(\int_0^\infty \frac{e^{-t} - e^{-\theta t}}{t} \mathbb{P}\{X_t \geq 0\} dt\right) \\ &= c\theta \exp\left(\int_0^\infty \frac{e^{-\theta t} - e^{-t}}{t} \mathbb{P}\{X_t < 0\} dt\right), \quad \theta > 0 \end{aligned} \quad (2.8)$$

where $c > 0$ is a constant and the second equality follows from Frullani's identity [13, Lemma 1.7]. Landau's theorem for Laplace transforms [16, Theorem II.5b] (and the monotone convergence theorem if $a = R$) imply that

$$\log \mathbb{E}[e^{aL_1^{-1}}] = ca \exp\left(\int_0^\infty \frac{e^{at} - e^{-t}}{t} \mathbb{P}\{X_t < 0\} dt\right) < \infty, \quad (2.9)$$

since the integral on the right-hand side is finite. Indeed, the convergence of the integral at $+\infty$ is guaranteed by (1.8), while the integrand remains bounded as $t \downarrow 0$.

Conversely, suppose that (1.7) holds. We claim that this ensures finiteness of the integral $\int_1^\infty t^{-1}\mathbb{P}\{X_t < 0\}dt$. Indeed, (1.7) comfortably implies $\mathbb{E}[L_1^{-1}] < \infty$. On the other hand, $\mathbb{E}[L_1^{-1}] = \lim_{\theta \rightarrow 0} \theta^{-1}(-\log \mathbb{E}[e^{-\theta L_1^{-1}}])$. Now use (2.8), which is still valid in the present situation, together with Fatou's lemma to conclude that

$$\int_0^\infty t^{-1}(1 - e^{-t})\mathbb{P}\{X_t < 0\} dt < \infty.$$

In particular, $\int_1^\infty t^{-1} \mathbb{P}\{X_t < 0\} dt < \infty$ as claimed. Consequently, $c' := c \exp(\int_0^\infty (1 - e^{-t}) t^{-1} \mathbb{P}\{X_t < 0\} dt)$ is finite and, therefore, we can rewrite (2.8) in the following form

$$-\log \mathbb{E}[e^{-\theta L_1^{-1}}] = c' \theta \exp \left(\int_0^\infty (e^{-\theta t} - 1) \frac{\mathbb{P}\{X_t < 0\}}{t} dt \right), \quad \theta \geq 0. \quad (2.10)$$

Hence, $\psi(\theta) := -\log \mathbb{E}[e^{-\theta L_1^{-1}}]/(c'\theta)$, $\theta > 0$ is the Laplace transform of an infinitely divisible law with Lévy measure $\nu(dt) = t^{-1} \mathbb{P}\{X_t < 0\} \mathbb{1}_{(0, \infty)}(t) dt$. Since $\mathbb{E}[e^{aL_1^{-1}}] < \infty$, ψ extends to a holomorphic function on a neighborhood of $(-a, 0]$ (notice that $\psi(0)$ is well-defined since $-\log \mathbb{E}[e^{-\theta L_1^{-1}}]$ has a zero of first or higher order at 0) and further extends continuously to the point $-a$. By Landau's theorem for Laplace transforms [16, Theorem II.5b], the Laplace transform on the right-hand side of (2.10) extends to $\text{Re}(\theta) > -a$ and, by the monotone convergence theorem, to $\theta = -a$. According to Theorem 25.17 in [14], this implies $\int_1^\infty e^{at} t^{-1} \mathbb{P}\{X_t < 0\} dt = \int_{(1, \infty)} e^{at} \nu(dt) < \infty$. By assumption, $\mathbb{P}\{X_1 < -\epsilon\} > 0$ for some $\epsilon > 0$. Therefore,

$$\int_1^\infty \frac{e^{at}}{t} \mathbb{P}\{X_t < 0\} dt \geq \frac{e^a}{2} \mathbb{P}\{X_1 < -\epsilon\} \int_1^\infty \frac{e^{at}}{t} \mathbb{P}\{X_t \leq \epsilon\} dt,$$

that is, (1.8) holds for $r = \epsilon$. \square

Proof of Theorem 1.3. The equivalences between (1.15), (1.16) and (1.17) have been established in [12, Theorem 1.3] and [11, Theorem 2.1(a)], respectively. Notice that in the cited references, the statements are formulated for nonnegative r only. However, the extension to $r \in \mathbb{R}$ is straightforward.

To prove the equivalence of (1.14) and (1.16), recall the definition of $I_k = \inf_{k \leq t \leq k+1} (X_t - X_k)$ and $S_k = \sup_{k \leq t \leq k+1} (X_t - X_k)$, $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, I_k and S_k are independent of X_{k-1} and have the same distributions as I_1 and S_1 , respectively. For $t > 0$ and $k \in \mathbb{N}_0$ such that $k \leq t \leq k+1$, we have

$$e^{ak} \mathbb{P}\{X_k + S_{k+1} \leq r\} \leq e^{at} \mathbb{P}\{X_t \leq r\} \leq e^{a(k+1)} \mathbb{P}\{X_k + I_{k+1} \leq r\}.$$

Integration over $t \in (0, \infty)$ leads to

$$\mathbb{E}[U_a^1(r - S_1)] \leq U_a(r) \leq e^a \mathbb{E}[U_a^1(r - I_1)].$$

Now assume that (1.14) holds. Then $U_a^1(r - x) < \infty$ for all $x > 0$ with $\mathbb{P}\{S_1 \leq x\} > 0$. According to the equivalence between (1.16) and (1.17), this implies $U_a(s) < \infty$ for all $s \in \mathbb{R}$. Conversely, when (1.16) holds, then, by the equivalence between (1.16) and (1.17), we have $U_a^1(r) < \infty$ for all $r \in \mathbb{R}$. Also, $a \leq R$ and thus there is a minimal $\gamma > 0$ with $\mathbb{E}[e^{-\gamma X_1}] = e^{-a}$. Further, for some $C > 0$, $\mathbb{E}[U_a^1(x)] \leq C e^{\gamma x}$ for all $x \geq 0$ (by Proposition 5.1 in [2]). From Lemma A.1 in the appendix, we infer $\mathbb{E}[e^{-\gamma I_1}] < \infty$ and, therefore,

$$U_a(r) \leq e^a \mathbb{E}[U_a^1(r - I_1)] \leq e^a C e^{\gamma r} \mathbb{E}[e^{-\gamma I_1}] < \infty. \quad (2.11)$$

To prove that (1.13) implies (1.14) first notice that by (2.3) we have

$$\int_0^\infty e^{at} \mathbb{P}\{\varrho_r > t\} dt \geq \int_0^\infty e^{at} \mathbb{P}\left\{\inf_{s \geq t} X_s < r\right\} dt \geq \int_0^\infty e^{at} \mathbb{P}\{X_t < r\} dt.$$

From previously established facts we conclude that the convergence of the last integral for some $r \in \mathbb{R}$ implies convergence of the integral for all $r \in \mathbb{R}$.

For the converse implication, assume that (1.14) holds, that is, $U_a(r) := \int_0^\infty e^{at} \mathbb{P}\{X_t \leq r\} dt < \infty$ for some $r \in \mathbb{R}$. According to the already proved equivalence between (1.16) and (1.14), $U_a(r) < \infty$ for all $r \in \mathbb{R}$. As in the proof of the equivalence between (1.14) and (1.16) and (2.11) we conclude that $U_a(r) \leq Ce^{\gamma r}$ for all $r \geq 0$ and some constant $C > 0$ where γ is the minimal positive root of the equation $\mathbb{E}[e^{-\gamma X_1}] = e^{-a}$. By (2.3), for $r \geq 0$,

$$\begin{aligned} \int_0^\infty e^{at} \mathbb{P}\{\varrho_r > t\} dt &\leq \int_0^\infty e^{at} \mathbb{P}\{X_t + \inf_{s \geq 0} (X_{t+s} - X_t) \leq r\} dt \\ &= \mathbb{E}\left[U_a\left(r - \inf_{s \geq 0} (X_{t+s} - X_t)\right)\right] = \mathbb{E}\left[U_a\left(r - \inf_{t \geq 0} X_t\right)\right] \\ &\leq Ce^{\gamma r} \mathbb{E}[e^{-\gamma \inf_{t \geq 0} X_t}] < \infty \end{aligned}$$

where the last inequality follows from Lemma A.1(b) which applies because $\mathbb{E}[e^{-\gamma X_1}] = e^{-a} < 1$. \square

Remark 2.1. In this remark, we briefly sketch another method that can be used to prove Theorems 1.2 and 1.3, namely, by drawing a connection to perturbed random walks. For instance, in order to see that (1.10) implies (1.6), define $I_n := \inf_{n-1 \leq t \leq n} (X_t - X_{n-1})$. Then $(X_1, I_1), (X_2 - X_1, I_2), \dots$ are i.i.d. 2-dimensional random vectors and $(\underline{X}_n)_{n \geq 0}$ with $\underline{X}_0 := 0$ and $\underline{X}_n := X_{n-1} + I_n$, $n \in \mathbb{N}$ is a perturbed random walk in the sense of [2]. If $\mathbb{E}[e^{a\underline{N}_r}] < \infty$ for some $r \geq 0$, then $0 < a \leq R$ and, in particular, $-\varphi'(0) = \mathbb{E}[X_1] \in (0, \infty]$. Hence $\lim_{t \rightarrow \infty} X_t = \infty$ a.s. by [14, Theorem 35.5]. Consequently, $\lim_{n \rightarrow \infty} \underline{X}_n = +\infty$ a.s. Therefore, we can conclude from [2, Theorem 2.6] that $\mathbb{E}[e^{a\underline{N}_r}] < \infty$ where $\underline{N}_r = \sum_{n \geq 0} \mathbb{1}_{\{\underline{X}_n \leq r\}}$. This implies (1.6) since

$$N_r = \int_0^\infty \mathbb{1}_{\{X_t \leq r\}} dt \leq \sum_{n \geq 0} \mathbb{1}_{\{\inf_{n \leq t \leq n+1} X_t \leq r\}} = 1 + \underline{N}_r.$$

Analogously, one can see that (1.15) implies (1.13) by using the same perturbed random walk $(\underline{X}_n)_{n \in \mathbb{N}_0}$ and the inequality $\varrho_r \leq \underline{\rho}(r)$ with $\underline{\rho}(r) := \sup\{n \in \mathbb{N}_0 : \underline{X}_n \leq r\}$. Indeed, $\mathbb{E}[e^{a\underline{\rho}(r)}] < \infty$ follows from (1.15), [2, Theorem 2.7(b)] and $\mathbb{E}[e^{-\gamma I_1}] < \infty$, which is contained in Lemma A.1(b).

Recall that in the proofs of Theorems 1.5 and 1.6, we exclude the case that X is a compound Poisson process.

Further, notice that (1.22) extends to a.s. finite \mathcal{F} -stopping times T (cf. [13, Corollary 3.11])

$$\left. \frac{d\mathbb{P}^\gamma}{d\mathbb{P}} \right|_{\mathcal{F}_T} = e^{-\gamma X_T + aT}. \quad (2.12)$$

For $T = T_r$ (which is finite a.s. in the situation considered here), this yields

$$\mathbb{E}[e^{aT_r}] = \mathbb{E}^\gamma[e^{\gamma X_{T_r}}] \quad \text{for } r > 0. \quad (2.13)$$

Proof of Theorem 1.5. Assume that the equivalent conditions of Theorem 1.2 hold.

(a) For each $t \geq 0$, set $H_t := X_{L_t^{-1}}$. Recall that, under \mathbb{P} , $(L_t^{-1}, H_t)_{t \geq 0}$ is a two-dimensional subordinator (without killing). Further, note that $\mathbb{E}[e^{aL_1^{-1}}] \in (1, \infty)$ by Theorem 1.2. Since L_1^{-1} and T_r are \mathcal{F} -stopping times, (2.12) gives

$$\mathbb{E}^\gamma[e^{\gamma H_1}] = \mathbb{E}[e^{aL_1^{-1}}] \in (1, \infty) \quad (2.14)$$

and

$$\mathbb{E}[e^{aT_r}] = \mathbb{E}^\gamma[e^{\gamma X_{T_r}}] = \mathbb{E}^\gamma[e^{\gamma H_{\tau_r}}] < \infty, \quad r > 0, \quad (2.15)$$

where $\tau_r := \inf\{t \geq 0 : H_t > r\}$. The second equality is a consequence of $X_{T_r} = H_{\tau_r}$ \mathbb{P}^γ -a.s. Under \mathbb{P}^γ , $(H_t)_{t \geq 0}$ is still a subordinator. Furthermore, (2.14) entails $\mathbb{E}^\gamma[H_1] < \infty$ which, in turn, implies that under \mathbb{P}^γ , $(H_t)_{t \geq 0}$ is a subordinator without killing.

If X is spectrally negative, then $X_{T_r} = r$ under \mathbb{P}^γ in which case $\mathbb{E}[e^{aT_r}] = e^{\gamma r}$. In Section 3 it is shown that the result in this particular case fits the general asymptotics stated in the theorem. In what follows we assume that X is not spectrally negative. Let H be a random variable with distribution

$$\mathbb{P}^\gamma\{H \in dx\} = \frac{d_\gamma}{\mathbb{E}^\gamma[H_1]} \delta_0(dx) + \frac{1}{\mathbb{E}^\gamma[H_1]} \Pi^\gamma((x, \infty)) \mathbb{1}_{(0, \infty)}(x) dx$$

where d_γ and Π^γ are the drift and the Lévy measure of the (infinitely divisible) \mathbb{P}^γ -law of H_1 . Then

$$\begin{aligned} \mathbb{E}^\gamma[e^{\gamma H}] &= \frac{d_\gamma}{\mathbb{E}^\gamma[H_1]} + \frac{1}{\mathbb{E}^\gamma[H_1]} \int_0^\infty e^{\gamma x} \Pi^\gamma((x, \infty)) dx \\ &= \frac{1}{\gamma \mathbb{E}^\gamma[H_1]} \left(d_\gamma \gamma + \int_{(0, \infty)} (e^{\gamma x} - 1) \Pi^\gamma(dx) \right) = \frac{\log \mathbb{E}^\gamma[e^{\gamma H_1}]}{\gamma \mathbb{E}^\gamma[H_1]} < \infty, \end{aligned} \quad (2.16)$$

where the finiteness follows from (2.14). Recalling (2.15) we conclude that it remains to prove that

$$\lim_{r \rightarrow \infty} \mathbb{E}^\gamma[e^{\gamma(H_{\tau_r} - r)}] = \mathbb{E}^\gamma[e^{\gamma H}].$$

To this end, we shall use the identity

$$\mathbb{P}\{H_{\tau_r} - r > s\} = \int_{(s, r+s]} (U_\gamma(r) - U_\gamma(r+s-t)) \Pi^\gamma(dt) + U_\gamma(r) \Pi^\gamma((r+s, \infty))$$

for $s \geq 0$ where $U_\gamma(dx) = \int_0^\infty \mathbb{P}^\gamma\{H_t \in dx\} dt$ and the relation

$$\begin{aligned} \lim_{r \rightarrow \infty} \mathbb{P}\{H_{\tau_r} - r > s\} &= \lim_{r \rightarrow \infty} \int_{(s, r+s]} (U_\gamma(r) - U_\gamma(r+s-t)) \Pi^\gamma(dt) \\ &= \mathbb{P}^\gamma\{H > s\} \end{aligned}$$

for $s \geq 0$, both of which can be found in the proof of Theorem 1 in [6]. With these at hand, we write

$$\begin{aligned}
\gamma^{-1} \mathbb{E}^\gamma [e^{\gamma(H_{\tau_r} - r)} - 1] &= \int_0^\infty e^{\gamma s} \mathbb{P}^\gamma \{H_{\tau_r} - r > s\} ds \\
&= \int_0^\infty e^{\gamma s} \int_{(s, r+s]} (U_\gamma(r) - U_\gamma(r + s - t)) \Pi^\gamma(dt) ds \\
&\quad + U_\gamma(r) \int_0^\infty e^{\gamma s} \Pi^\gamma((r + s, \infty)) ds \\
&=: A(r) + B(r).
\end{aligned} \tag{2.17}$$

Now we use the representation

$$B(r) = e^{-\gamma r} U_\gamma(r) \int_r^\infty e^{\gamma y} \Pi^\gamma((y, \infty)) dy$$

to infer $\lim_{r \rightarrow \infty} B(r) = 0$ because $\lim_{r \rightarrow \infty} \int_r^\infty e^{\gamma y} \Pi^\gamma((y, \infty)) dy = 0$ which follows from (2.16), and $\lim_{r \rightarrow \infty} r^{-1} U_\gamma(r) = (\mathbb{E}^\gamma[H_1])^{-1}$. Using subadditivity of U_γ we have

$$\int_{(s, r+s]} (U_\gamma(r) - U_\gamma(r + s - t)) \Pi^\gamma(dt) \leq \int_{(s, \infty)} U_\gamma(t - s) \Pi^\gamma(dt).$$

Furthermore,

$$\begin{aligned}
&\int_0^\infty e^{\gamma s} \int_{(s, \infty)} U_\gamma(t - s) \Pi^\gamma(dt) ds \\
&= \int_{(0, \infty)} e^{\gamma t} \int_0^t e^{-\gamma y} U_\gamma(y) dy \Pi^\gamma(dt) \\
&\leq U_\gamma(1) \int_{(0, 1)} t e^{\gamma t} \Pi^\gamma(dt) + \int_0^\infty e^{-\gamma y} U_\gamma(y) dy \int_{[1, \infty)} e^{\gamma t} \Pi^\gamma(dt) < \infty.
\end{aligned}$$

Here, the first integral in the last line is finite because Π^γ is a Lévy measure which must satisfy $\int_{(0, 1)} t \Pi^\gamma(dt) < \infty$. The finiteness of the second integral follows from the fact that $\lim_{r \rightarrow \infty} r^{-1} U_\gamma(r) = (\mathbb{E}^\gamma[H_1])^{-1}$, while the finiteness of the third integral follows from (2.16). Therefore,

$$\lim_{r \rightarrow \infty} A(r) = \int_0^\infty e^{\gamma s} \mathbb{P}^\gamma \{H > s\} ds = \gamma^{-1} \mathbb{E}^\gamma [e^{\gamma H} - 1]$$

by the dominated convergence theorem. In view of (2.17) the proof is complete.

(b) For $r \in \mathbb{R}$, set $f(r) := \mathbb{E}[e^{a N_r}]$ and $g(r) := e^{\gamma r} f(-r)$. Using the decomposition

$$N_r = T_r + \int_{T_r}^\infty \mathbb{1}_{\{X_t - X_{T_r} \leq r - X_{T_r}\}} dt$$

and recalling that $\int_{T_r}^\infty \mathbb{1}_{\{X_t - X_{T_r} \leq s\}} dt$ is independent of (T_r, X_{T_r}) and has the same law as N_s we infer

$$f(r) = \mathbb{E}[e^{a T_r} f(r - X_{T_r})] = e^{\gamma r} \mathbb{E}^\gamma [g(X_{T_r} - r)].$$

If X is spectrally negative, then $f(r) = e^{\gamma r} \mathbb{E}[e^{aN_0}]$. Suppose X is not spectrally negative. Then $g(X_{T_r} - r) \leq e^{\gamma(X_{T_r} - r)} f(0)$ a.s. From the proof of part (a), we know that $\lim_{r \rightarrow \infty} e^{\gamma(X_{T_r} - r)} f(0) = e^{\gamma H} f(0)$ in \mathbb{P}^γ -distribution and $\lim_{r \rightarrow \infty} \mathbb{E}[e^{\gamma(X_{T_r} - r)}] f(0) = \mathbb{E}[e^{\gamma H}] f(0)$. The function f is nondecreasing, hence it has countably many discontinuities. Since the law of H under \mathbb{P}^γ is absolutely continuous on $(0, \infty)$ we have $\lim_{r \rightarrow \infty} g(X_{T_r} - r) = g(H)$ in \mathbb{P}^γ -distribution. Now the desired conclusion $\lim_{r \rightarrow \infty} \mathbb{E}^\gamma[g(X_{T_r} - r)] = \mathbb{E}^\gamma[g(H)]$ follows from Pratt's lemma which is a (slightly more general) version of the dominated convergence theorem. \square

Proof of Theorem 1.6. Assume that the equivalent conditions in Theorem 1.3 hold, in particular, $U_a(r) < \infty$ for every $r \in \mathbb{R}$. Furthermore, either $a \in (0, R)$ or $a = R$ and $\mathbb{E}[X_1 e^{-\gamma X_1}] > 0$. We shall use the probability measure \mathbb{P}^γ defined in (1.22). In view of (1.23) and the discussion following it we have

$$\nu_\gamma := \mathbb{E}^\gamma[X_1] = e^a \mathbb{E}[X_1 e^{-\gamma X_1}] \in (0, \infty). \quad (2.18)$$

For $r \in \mathbb{R}$, we write $U_a(r)$ in the following form

$$U_a(r) = \int_0^\infty \mathbb{E}^\gamma[e^{\gamma X_t} \mathbb{1}_{\{X_t \leq r\}}] dt = \int_{(-\infty, r]} e^{\gamma x} U^\gamma(dx) \quad (2.19)$$

where $U^\gamma(dx) := \int_0^\infty \mathbb{P}^\gamma\{X_t \in dx\} dt$ denotes the potential measure of X under \mathbb{P}^γ . It is well-known (and can be checked by a simple calculation) that $U^\gamma = Z^\gamma - \delta_0$ where δ_0 is the Dirac measure with mass 1 at the point 0 and $Z^\gamma(\cdot) := \sum_{n \geq 0} \mathbb{P}^\gamma\{X_{\tau_n} \in \cdot\}$ is the renewal measure of the zero-delayed random walk $(X_{\tau_n})_{n \in \mathbb{N}_0}$ where $\tau_0 := 0$ and $(\tau_n - \tau_{n-1})_{n \in \mathbb{N}}$ is a sequence of i.i.d. exponential random variables with unit mean independent of X . In particular,

$$\mathbb{P}^\gamma\{X_{\tau_1} \in \cdot\} = \int_0^\infty e^{-t} \mathbb{P}^\gamma\{X_t \in \cdot\} dt.$$

Observe that $\mathbb{E}^\gamma[X_{\tau_1}] = \nu_\gamma$. From this it is clear that the asymptotic behavior of $U_a(r)$ as $r \rightarrow \infty$ coincides with that of $\int_{(-\infty, r]} e^{\gamma x} Z^\gamma(dx)$.

Since we exclude the case that X is a compound Poisson process, the distribution of X_{τ_1} under \mathbb{P}^γ is non-arithmetic. Further, the function $x \mapsto e^{-\gamma x} \mathbb{1}_{[0, \infty)}(x)$ is directly Riemann integrable. We can, therefore, invoke the key renewal theorem on the whole line to conclude that

$$\begin{aligned} e^{-\gamma r} \int_{(-\infty, r]} e^{\gamma x} Z^\gamma(dx) &= \int e^{-\gamma(r-x)} \mathbb{1}_{[0, \infty)}(r-x) Z^\gamma(dx) \\ &\xrightarrow{r \rightarrow \infty} \frac{1}{\nu_\gamma} \int_0^\infty e^{-\gamma x} dx = \frac{1}{\gamma \nu_\gamma}, \end{aligned}$$

where we have used $\nu_\gamma > 0$. This in combination with (2.18) implies (1.26).

Regarding (1.27), we use (2.3) for $r \geq 0$ to conclude that

$$\begin{aligned} \frac{1}{a} \mathbb{E}[e^{a\varrho_r} - 1] &= \int_0^\infty e^{at} \mathbb{P}\{\varrho_r > t\} dt \leq \int_0^\infty e^{at} \mathbb{P}\{\inf_{s \geq t} X_s \leq r\} dt \\ &= \int_0^\infty e^{at} \mathbb{P}\{X_t + I'_t \leq r\} dt = \mathbb{E}[U_a(r - I)] \end{aligned}$$

where $I'_t := \inf_{s \geq t} (X_s - X_t)$ has the same law as $I = \inf_{s \geq 0} X_s$ and is independent of X_t . Similarly, using the lower bound provided by (2.3), we get

$$\frac{1}{a} \mathbb{E}[e^{a\varrho_r} - 1] \geq \int_0^\infty e^{at} \mathbb{P}\{\inf_{s \geq t} X_s < r\} dt = \mathbb{E}[U_a((r - I)-)]$$

where $U_a(s-) := \sum_{n \geq 0} e^{an} \mathbb{P}\{X_n < s\}$, $s \in \mathbb{R}$. The argument used above that reveals the asymptotic behavior of $U_a(s)$ as $s \rightarrow \infty$ also shows that $U_a(s-)$ exhibits the same asymptotic behavior. Now (1.27) follows from (1.26) from the dominated convergence theorem and $\mathbb{E}[e^{-\gamma I}] < \infty$ (see Lemma A.1). \square

3 Particular cases and examples

We begin the section with the proof of Remark 1.4.

Proof of Remark 1.4. Observe that

$$\mathbb{P}\{T_r > t\} = \sum_{n \geq 0} \mathbb{P}\{N(t) = n\} \mathbb{P}\{\tau(r) > n\}.$$

Multiplying by e^{at} and integrating w.r.t. t gives

$$\begin{aligned} \frac{1}{a} \mathbb{E}[e^{aT_r} - 1] &= \int_0^\infty e^{at} \mathbb{P}\{T_r > t\} dt \\ &= \sum_{n \geq 0} \int_0^\infty \frac{(\lambda t)^n}{n!} e^{-\lambda t} e^{at} dt \mathbb{P}\{\tau(r) > n\} \\ &= \frac{1}{\lambda - a} \sum_{n \geq 0} \left(\frac{\lambda}{\lambda - a}\right)^n \mathbb{P}\{\tau(r) > n\} \\ &= \frac{1}{\lambda - a} \sum_{n \geq 0} e^{bn} \mathbb{P}\{\tau(r) > n\} \\ &= \frac{1}{a} \mathbb{E}[e^{b\tau(r)} - 1] \end{aligned}$$

where we used Fubini's theorem for nonnegative integrands to interchange summation and integration. Fubini's theorem in particular implies that the left-hand side is finite if and only if the right hand side is.

The observation for N_r is even simpler: Note that for any n with $S_n \leq r$, the corresponding compound Poisson process spends an exponentially distributed time (independent of $(S_n)_{n \in \mathbb{N}_0}$) below the level r . Therefore, with $(e_k)_{k \in \mathbb{N}}$ denoting a sequence of i.i.d. exponentials with mean $1/\lambda$ which is independent of $n(r)$, we have

$$N_r \stackrel{d}{=} \sum_{k=1}^{n(r)} e_k.$$

From this we readily derive

$$\mathbb{E}[e^{aN_r}] = \mathbb{E}\left[\left(\frac{\lambda}{\lambda - a}\right)^{n(r)}\right] = \mathbb{E}[e^{bn(r)}].$$

For the relation between ϱ_r and $\rho(r)$ one proceeds as for T_r :

$$\mathbb{P}\{\varrho_r > t\} = \sum_{n \geq 0} \mathbb{P}\{N(t) = n\} \mathbb{P}\{\rho(r) > n\},$$

from which the desired relation follows. \square

Example 3.1 (Spectrally negative Lévy processes). Let X be spectrally negative, $0 < a \leq R = -\log \inf_{t \geq 0} \varphi(t)$ and γ as in (1.21). Then

$$\mathbb{E}[e^{aT_r}] = e^{\gamma r}, \quad r \geq 0, \quad (3.1)$$

$$\mathbb{E}[e^{aN_r}] = e^{\gamma r} \gamma a^{-1} \mathbb{E}[X_1], \quad r \geq 0, \quad (3.2)$$

$$\mathbb{E}[e^{a\varrho_r}] = e^{\gamma r} \frac{e^{-a} \mathbb{E}[X_1]}{\mathbb{E}[X_1 e^{-\gamma X_1}]}, \quad r \geq 0 \quad (3.3)$$

where the last relation holds whenever $a \in (0, R)$ or $a = R$ and $\mathbb{E}[X_1 e^{-\gamma X_1}] > 0$.

Before we prove relations (3.1), (3.2) and (3.3), notice that since X is spectrally negative we have $L_r^{-1} = T_r$ and $X_{L_r^{-1}} = r$ \mathbb{P} -a.s. and \mathbb{P}^γ -a.s. Hence using (3.1) we infer

$$\frac{\log \mathbb{E}[e^{aL_1^{-1}}]}{\gamma \mathbb{E}^\gamma[X_{L_1^{-1}}]} = \frac{\log \mathbb{E}[e^{aT_1}]}{\gamma} = 1.$$

This shows that (3.1) is in full agreement with (1.24).

Using (A.1) we further see that the asymptotic behavior in (3.3) agrees with that in (1.27).

Proof of (3.1). Since X is spectrally negative, $X_{T_r} = r$ \mathbb{P} -a.s. and \mathbb{P}^γ -a.s. for all $r > 0$. Hence (3.1) is a consequence of (2.13). \square

In the proof of (3.3), we make explicit the dependence of γ on a . To be more precise, let γ_0 denote the unique positive real with $\varphi(\gamma_0) = e^{-R}$ where $R = -\log \inf_{\theta \geq 0} \varphi(\theta)$. Then $-\log \varphi : [0, \gamma_0] \rightarrow [0, R]$ is a bijection. Let $\gamma : [0, R] \rightarrow [0, \gamma_0]$ denote its inverse, so $\varphi(\gamma(a)) = e^{-a}$. Note for later use that differentiating the latter relation with respect to a and solving for $\gamma'(a)$ gives

$$\gamma'(a) = \frac{e^{-a}}{-\varphi'(\gamma(a))} = \frac{e^{-a}}{\mathbb{E}[X_1 e^{-\gamma(a)X_1}]} \quad (3.4)$$

for all a for which $\mathbb{E}[X_1 e^{-\gamma(a)X_1}]$ is finite and positive. The set of these a includes the interval $(0, R)$ and, additionally, the point R when (1.17) holds.

Proof of (3.3). Observe that using spectral negativity, for $r \geq 0$,

$$\mathbb{E}[e^{a\varrho_r}] = \mathbb{E}[e^{aT_r}] \mathbb{E}[e^{a\varrho_0}]. \quad (3.5)$$

To see this, it suffices to decompose ϱ_r in the following form

$$\begin{aligned} \varrho_r &= T_r + \sup\{t \geq 0 : X_{T_r+t} - X_{T_r} + X_{T_r} \leq r\} \\ &= T_r + \sup\{t \geq 0 : X_{T_r+t} - X_{T_r} \leq 0\} \end{aligned}$$

and to note that the second term is independent of T_r and has the same law as ϱ_0 . Further we claim that since X is spectrally negative, for $r \geq 0$,

$$\mathbb{P}\{\varrho_r > t\} \leq \mathbb{P}\{\inf_{s \geq t} X_s \leq r\} \leq \mathbb{P}\{\varrho_r \geq t\} \quad (3.6)$$

for all $t \geq 0$. Indeed, $\mathbb{P}\{\varrho_r > t\} \leq \mathbb{P}\{\inf_{s \geq t} X_s \leq r\}$ by (2.3). To see that the second inequality holds, it is enough to show that $\mathbb{P}\{\inf_{s \geq t} X_s \leq r, \varrho_r < t\} = 0$ for all $t \geq 0$. The latter follows from the fact that when $\inf_{s \geq t} X_s \leq r$, then there is an $s \geq t$ with $X_s \leq r$ which implies $\varrho_r \geq s$ or there is a sequence $(s_n)_{n \in \mathbb{N}}$ with $s_n \geq t$ and $X_{s_n} > r$ for all $n \in \mathbb{N}$, but $\lim_{n \rightarrow \infty} X_{s_n} = r$. We can assume without loss of generality that $(s_n)_{n \in \mathbb{N}}$ is monotone and hence $s := \lim_{n \rightarrow \infty} s_n$ exists in $[t, \infty)$ ($s < \infty$ since in the given situation, X drifts to $+\infty$ a.s.). If $(s_n)_{n \in \mathbb{N}}$ is decreasing, then, by the right-continuity of the paths, $X_s = r$ and we are in the first case. If $(s_n)_{n \in \mathbb{N}}$ is increasing, then again $X_s \leq r$ by the absence of positive jumps. Consequently, since the probabilities on the left and right of (3.6) coincide for all but countably many t ,

$$\begin{aligned} \frac{1}{a} \mathbb{E}[e^{a\varrho_r} - 1] &= \int_0^\infty e^{at} \mathbb{P}\{\inf_{s \geq t} X_s \leq r\} dt \\ &= \int_0^\infty e^{at} \mathbb{P}\{X_t < 0\} dt + \int_0^\infty e^{at} \int_{[0, \infty)} \mathbb{P}\{\inf_{s \geq 0} X_s \leq r - x\} \mathbb{P}\{X_t \in dx\} dt \\ &=: I(a) + J(r). \end{aligned}$$

To calculate $J(r)$ we make essential use of the identity

$$t \mathbb{P}\{T_x \in dt\} dx = x \mathbb{P}\{X_t \in dx\} dt, \quad (x, t) \in [0, \infty) \times [0, \infty),$$

see [5, Corollary VII.3]. With this notation,

$$\begin{aligned} J(r) &= \int_0^\infty x^{-1} \mathbb{P}\{\inf_{s \geq 0} X_s \leq r - x\} \int_{[0, \infty)} t e^{at} \mathbb{P}\{T_x \in dt\} dx \\ &= \int_0^\infty x^{-1} \mathbb{P}\{\inf_{s \geq 0} X_s \leq r - x\} \frac{\partial}{\partial a} \mathbb{E}[e^{aT_x}] dx \\ &= \gamma'(a) \int_0^\infty \mathbb{P}\{\inf_{s \geq 0} X_s \leq r - x\} e^{\gamma(a)x} dx \\ &= \gamma'(a) \mathbb{E} \left[\int_0^{r - \inf_{s \geq 0} X_s} e^{\gamma(a)x} dx \right] = \frac{\gamma'(a)}{\gamma(a)} (e^{\gamma(a)r} \mathbb{E}[e^{-\gamma(a) \inf_{s \geq 0} X_s}] - 1) \end{aligned}$$

having utilized (3.1) for the third equality. In view of (3.5), we have

$$\begin{aligned} I(a) &= \frac{1}{a} \mathbb{E}[e^{a\varrho_r} - 1] - J(r) \\ &= \frac{1}{a} (e^{\gamma(a)r} \mathbb{E}[e^{\varrho_0}] - 1) - \frac{\gamma'(a)}{\gamma(a)} (e^{\gamma(a)r} \mathbb{E}[e^{-\gamma(a) \inf_{s \geq 0} X_s}] - 1) \\ &= \frac{\gamma'(a)}{\gamma(a)} - \frac{1}{a} \end{aligned} \quad (3.7)$$

since $I(a)$ does not depend on r . Further,

$$\mathbb{E}[e^{a\varrho_r}] = e^{\gamma(a)r} \frac{a\gamma'(a)}{\gamma(a)} \mathbb{E}[e^{-\gamma(a) \inf_{s \geq 0} X_s}].$$

According to (A.1) and (3.4), we have $\mathbb{E}[e^{-\gamma(a) \inf_{s \geq 0} X_s}] = \gamma(a)\mathbb{E}[X_1]/a$ and $\gamma'(a) = e^{-a}/\mathbb{E}[X_1 e^{-\gamma(a)X_1}]$ which completes the proof of (3.3). \square

Proof of (3.2). The same argument as for (3.5) yields

$$\mathbb{E}[e^{aN_r}] = \mathbb{E}[e^{aT_r}]\mathbb{E}[e^{aN_0}] \quad (3.8)$$

for $a \in [0, R]$. Letting $f(a) := \mathbb{E}[e^{aN_0}]$, taking logarithms on both sides of (2.7) and then differentiating with respect to $a \in (0, R]$, we infer

$$(\log f(a))' = \int_0^\infty e^{at} \mathbb{P}\{X_t \leq 0\} dt = I(a) = (\log \gamma(a) - \log a)'$$

having used (3.7) for the last equality. Hence $f(a) = c\gamma(a)/a$ for some constant $c > 0$. $f(0) = 1$ and $\lim_{a \downarrow 0} \gamma(a)/a = \gamma'(0) = 1/\mathbb{E}[X_1]$ imply $c = \mathbb{E}[X_1]$. \square

Example 3.2 (Stable subordinators). Let X be an α -stable subordinator, $\alpha \in (0, 1)$ with Laplace exponent $\Psi(-\theta) = -\theta^\alpha$, $\theta \geq 0$. The process $(T_r)_{r \geq 0}$ is called an inverse α -stable subordinator. It is well known (see [7, Proposition 1(a)]) that T_r has a Mittag-Leffler distribution with moments $\mathbb{E}[T_r^n] = r^{n\alpha} n! / \Gamma(1 + n\alpha)$, $n \in \mathbb{N}_0$ where $\Gamma(\cdot)$ is the gamma function. Hence, for any $a \geq 0$ and $r \geq 0$

$$\mathbb{E}[e^{aT_r}] = \sum_{n \geq 0} \frac{(ar^\alpha)^n}{\Gamma(1 + n\alpha)} = E_\alpha(ar^\alpha) < \infty$$

where $E_\alpha(\cdot)$ denotes the Mittag-Leffler function with parameter α . Note that (by [?], p. 315) this is in accordance with the asymptotics stated in (1.24).

Example 3.3 (Brownian motion with drift). For $\mu > 0$ and a standard Brownian motion $(B_t)_{t \geq 0}$, let $X_t = \mu t + B_t$, $t \geq 0$. According to [14, Example 46.6], $(T_r)_{r \geq 0}$ is an inverse Gaussian subordinator with distribution

$$\mathbb{P}\{T_r \in dy\} = \frac{r e^{\mu r}}{\sqrt{2\pi}} e^{-\mu^2 y/2 - r^2/(2y)} y^{-3/2} \mathbb{1}_{(0, \infty)}(y) dy.$$

This implies $\mathbb{E}[e^{aT_r}] < \infty$ iff $a \leq \mu^2/2$. This is in full agreement with Theorem 1.2 because X_1 has Laplace exponent $-\log \varphi(\theta) = \mu\theta - \theta^2/2$, $\theta \geq 0$. This function attains its supremum at μ , hence $R = \sup_{\theta \geq 0} (\mu\theta - \theta^2/2) = \mu^2/2$. Finally, for $a \leq \mu^2/2$, in view of (3.1),

$$\mathbb{E}[e^{aT_r}] = e^{(\mu - \sqrt{\mu^2 - 2a})r}.$$

According to [8, Formula 1.5.4(1) on p. 204]

$$\mathbb{P}\{N_r \in dy\} = \left(\frac{\mu\sqrt{2}}{\sqrt{\pi}y} e^{-(r-\mu y)^2/(2y)} - \frac{2\mu^2 e^{2r\mu}}{\sqrt{\pi}} \int_s^\infty e^{-x^2} dx \right) \mathbb{1}_{(0, \infty)}(y) dy,$$

where $s = r/\sqrt{2y} + \mu\sqrt{y/2}$. We only give detailed calculations for the case $r = 0$ and denote the corresponding density by $f(y)$. Using

$$\frac{e^{-s^2}}{2s} - \int_s^\infty e^{-x^2} dx \sim \frac{e^{-s^2}}{4s^3} \quad \text{as } s \rightarrow \infty,$$

which can be obtained using L'Hôpital's rule, we infer

$$e^{\mu^2 y/2} f(y) \sim \text{const } y^{-3/2} \quad \text{as } y \rightarrow \infty.$$

Thus, $\mathbb{E}[e^{aN(0)}] < \infty$ iff $a \leq \mu^2/2 = R$ in agreement with Theorem 1.2.

Finally, for any $r \geq 0$, according to [8, Point 31 on p. 65]

$$\mathbb{P}\{\varrho_r \in dy\} = \frac{\mu}{\sqrt{2\pi y}} e^{\mu r - \mu^2 y/2 - r^2/(2y)} \mathbb{1}_{(0,\infty)}(y) dy.$$

Therefore, $\mathbb{E}[e^{a\varrho_r}] < \infty$ iff $a < \mu^2/2 = R$ which is in agreement with Theorem 1.3 because $\mathbb{E}[X_1 e^{-\mu X_1}] = 0$. Finally, a quick calculation (using the characteristic function of a Lévy distribution) shows that

$$\mathbb{E}[e^{a\varrho_r}] = \frac{\mu}{\sqrt{\mu^2 - 2a}} e^{(\mu - \sqrt{\mu^2 - 2a})r}$$

whenever $a < R$. This confirms (1.27).

A Auxiliary results

The results summarized in the following lemma should be known. We prove them because we have not been able to locate a proper reference.

Lemma A.1. *Define $I_t := \inf_{0 \leq s \leq t} X_s$ and $I := \inf_{t \geq 0} X_t$.*

- (a) *If $\mathbb{E}[e^{-\theta X_1}] < \infty$ for some $\theta > 0$, then $\mathbb{E}[e^{-\theta I_1}] < \infty$.*
- (b) *If $\mathbb{E}[e^{-\theta X_1}] < 1$ for some $\theta > 0$, then $\mathbb{E}[e^{-\theta I}] < \infty$. Furthermore, if X is spectrally negative, then*

$$\mathbb{E}[e^{-\theta I}] = \frac{\theta \mathbb{E}[X_1]}{-\log \mathbb{E}[e^{-\theta X_1}]}. \quad (\text{A.1})$$

Proof. (a) We first observe that $\mathbb{E}[e^{-\theta X_1}] < \infty$ entails $\mathbb{E}[e^{\theta X_1^-}] < \infty$. Now use the following inequality due to Willekens [17]

$$\mathbb{P}\{\sup_{0 \leq t \leq 1} (-X_t) \geq u\} \mathbb{P}\{\inf_{0 \leq t \leq 1} (-X_t) \geq -u_0\} \leq \mathbb{P}\{-X_1 \geq u - u_0\}$$

for $u_0 \in (0, u)$ to conclude that $\mathbb{E}[e^{-\theta I_1}] = \mathbb{E}[e^{\theta \sup_{0 \leq t \leq 1} (-X_t)}] < \infty$ follows from $\mathbb{E}[e^{\theta X_1^-}] < \infty$.

(b) Note that $\mathbb{E}[e^{-\theta X_1}] < 1$ for some $\theta > 0$ entails $\mathbb{E}[X_1] \in (0, \infty]$ and thus $\lim_{t \rightarrow \infty} X_t = +\infty$ a.s. Hence, $I = \inf_{t \geq 0} X_t$ is a.s. finite and, moreover, $\lim_{t \rightarrow \infty} \inf_{s \geq t} X_s = \lim_{t \rightarrow \infty} (X_t + I'_t) = +\infty$ a.s. where $I'_t = \inf_{s \geq t} (X_s - X_t)$. Observe further that

$$\exp(-\theta I) \leq \exp(-\theta I_1) + \exp(-\theta X_1) \exp(-\theta I'_1), \quad (\text{A.2})$$

Now write $I_{s:t} := \inf_{s \leq u \leq t} (X_u - X_s)$ and iterate (A.2) n times to obtain

$$\exp(-\theta I) \leq \sum_{k=0}^{n-1} \exp(-\theta X_k) \exp(-\theta I_{k:k+1}) + \exp(-\theta X_n) \exp(-\theta I'_n).$$

Letting $n \rightarrow \infty$ and using that $\exp(-\theta X_n) \exp(-\theta I'_n) = \exp(-\theta(X_n + I'_n)) \rightarrow 0$ a.s., one infers

$$\exp(-\theta I) \leq \sum_{k \geq 0} \exp(-\theta X_k) \exp(-\theta I_{k:k+1}).$$

$I_{k:k+1}$ is a copy of I_1 and independent of X_k for each $k \in \mathbb{N}_0$ and since $\mathbb{E}[e^{-\theta I_1}] < \infty$ by part (a) of the lemma we conclude that $\mathbb{E}[e^{-\theta I}] \leq \mathbb{E}[e^{-\theta I_1}](1 - \mathbb{E}[e^{-\theta X_1}])^{-1} < \infty$.

The Wiener-Hopf factorization (Theorem 45.2 and Theorem 45.7 in [14]) is equivalent to the distributional equalities

$$U_q \stackrel{d}{=} V_q + W_q$$

for $q > 0$ where V_q and W_q are independent, U_q has the same distribution as X_τ with τ denoting an exponential random variable with parameter q independent of X , V_q has the same distribution as S_τ (with $S_t := \sup_{0 \leq s \leq t} X_s$) and W_q has the same distribution as I_τ . We have $\mathbb{E}[e^{-\theta U_q}] = q(q - \log \varphi(\theta))$ and $\mathbb{E}[e^{-\theta V_q}] = \gamma^*(q)(\gamma^*(q) + \theta)^{-1}$ for all $\theta \geq 0$ where γ^* is the inverse of $\theta \mapsto \log \varphi(-\theta)$. The latter formula can be found in various sources, for instance, in the proof of Theorem 46.3 in [14]. Consequently,

$$\mathbb{E}[e^{-\theta W_q}] = \frac{q}{q - \log \varphi(\theta)} \frac{\gamma^*(q) + \theta}{\gamma^*(q)}, \quad \theta \geq 0.$$

Since $q/\gamma^*(q) \rightarrow \mathbb{E}[X_1]$ as $q \downarrow 0$, the right-hand side tends to the right-hand side of (A.1). Applying the monotone convergence theorem twice we conclude that

$$\mathbb{E}[e^{-\theta W_q}] = \int_0^\infty e^{-u} \mathbb{E}[e^{-\theta I_{u/q}}] du \rightarrow \mathbb{E}[e^{-\theta I}], \quad q \rightarrow 0.$$

□

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